Third-Order Differential Variational Principles and Differential Equations of Motion for Mechanico-Electrical Systems

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Abstract: This paper investigates the third-order differential variational principles of mechanico-electrical systems and the third-order differential equations of motion for holonomic mechanico-electrical systems. Based on Newton's laws of motion of mechanical systems and Kirchhoff's voltage laws of circuit systems, the third-order d'Alembert principles of mechanico-electrical systems are proposed, the third-order d'Alembert-Lagrange principle of mechanico-electrical systems is established and the parametric forms of Euler-Lagrange, Nielsen and Appell for this principle are given. Finally, the several different forms of the third-order differential equations of motion for holonomic mechanico-electrical systems are obtained.

1. Introduction

Mechanico-electrical systems, in which a mechanical process and an electromagnetic process are coupled to each other, play an important role in physics, mechanics and many fields of science and technology. Maxwell J C [1] described the mechanico-electrical systems by using Lagrange method, obtained the dynamical equations of the systems which are called the Lagrange-Maxwell equations afterwards. Mei et al. [2] established the Lagrange-Maxwell equations of discrete mechanico-electrical systems from the viewpoint of energy. Qiu [3] studied analytical dynamics of mechanico-electrical systems systematically, introduced the integral variational principles of electromagnetic systems and mechanical systems. Fu et al. [4] proposed the discrete variational principle and the first integrals for Lagrange-Maxwell mechanico-electrical systems. In recent years, some important results on the study of symmetries and conserved quantities of mechanico-electrical systems have also been obtained [5-16].

However, due to the change in operating conditions and the random disturbance of external load, the practical mechanico-electrical systems are often acted by variable mechanical forces and variable electromotive forces, so the research on the motion law of mechanico-electrical systems under the time rate of change of mechanical and electromotive force has important significance. Mei et al. [2] and Ma et al. [17-20] studied the third-order Lagrange equations for several different mechanical systems. Zhang [21-25] further studied the higher-order differential equations of motion for several different mechanical systems. But the researches on the third-order differential variational principles and the third-order differential equations of motion for mechanico-electrical systems have not been reported yet.

In this paper, starting from Newton's laws of motion for mechanical systems and Kirchhoff's voltage laws for circuit systems, the third-order d'Alembert principles of mechanico-electrical systems are proposed, the third-order d'Alembert-Lagrange principle of mechanico-electrical systems and its parametric forms will be established, and the third-order Lagrange-Maxwell equations, Nielsen equations, and Appell equations of holonomic mechanico-electrical systems will be obtained.

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2. Third-Order D'Alembert-Lagrange Principle of Mechanico-Electrical Systems

Suppose that the mechanical part of mechanico-electrical systems, constituted by N particles, is described by s generalized coordinates q_α $(\alpha = 1, 2, \cdots, s)$; the electromagnetic part of mechanico-electrical systems, constituted by $m+1$ non-independent loops (where m inner loops and one outer loop) with linear conductors and capacitors, is described by m generalized electric quantities e_j $(j = 1, 2, \cdots, m)$. Based on the Newton's laws of motion for the i-th particle and the Kirchhoff's voltage laws for the k-th non-independent loop respectively, we have

$$-m_i \ddot{\mathbf{r}}_i + \mathbf{F}_i + \mathbf{R}_i = 0 \quad (i = 1, 2, \cdots, N) \tag{1}$$

$$-\frac{d}{dt}\left(\sum_{l=1}^{m+1} L_{kl} \dot{E}_l\right) - R_k \dot{E}_k - \frac{E_k}{C_k} + U_k = 0 \quad (k = 1, 2, \cdots, m+1) \tag{2}$$

where m_i and $\mathbf{r}_i$ are, respectively, the mass and the position vector of i-th particle; $\mathbf{F}_i$ and $\mathbf{R}_i$ are, respectively, the active force and the constraint force acted on the i-th particle; L_{kl} $(l \neq k)$ and L_{kk} are, respectively, the mutual inductance between the k-th and l-th loops, and the self-inductance of the k-th loop; R_k and C_k are, respectively, the resistance and the capacitance of the k-th loops; E_k and $\dot{E}_k$ are, respectively, the capacitor charge and the current of the k-th loop (with $\dot{E}_k = I_k$); U_k is the electromotive force of k-th loop. The electromechanical analogy shows that the equation (2) in electromagnetic systems is equivalent to the equation (1) in mechanical systems, therefore, equations (1) and (2) can be called the d'Alembert principles of mechanico-electrical systems.

Differentiating Eqs.(1) and (2), we obtain

$$-m_i \dddot{\mathbf{r}}_i + \dot{\mathbf{F}}_i + \dot{\mathbf{R}}_i = 0 \quad (i = 1, 2, \cdots, N) \tag{3}$$

$$-\frac{d^2}{dt^2}\left(\sum_{l=1}^{m+1} L_{kl} \dot{E}_l\right) - R_k \ddot{E}_k - \frac{d}{dt}\left(\frac{E_k}{C_k}\right) + \dot{U}_k = 0 \quad (k = 1, 2, \cdots, m+1) \tag{4}$$

Equations (3) and (4) can be called the third-order d'Alembert principles of mechanico-electrical systems. Making the dot product with $\delta \mathbf{r}_i$ and summing over i for Eq.(3), multiplying by δE_k and summing over k for Eq.(4), and then adding them together, we have

$$\sum_{i=1}^{N}\left(-m_i \dddot{\mathbf{r}}_i + \dot{\mathbf{F}}_i\right) \cdot \delta \mathbf{r}_i + \sum_{k=1}^{m+1}\left[-\frac{d^2}{dt^2}\left(\sum_{l=1}^{m+1} L_{kl} \dot{E}_l\right) - R_k \ddot{E}_k - \frac{d}{dt}\left(\frac{E_k}{C_k}\right) + \dot{U}_k\right] \delta E_k = 0 \tag{5}$$

where we assume that the condition of ideal constraint is satisfied, namely

$$\sum_{i=1}^{N} \dot{\mathbf{R}}_i \cdot \delta \mathbf{r}_i = 0 \tag{6}$$

Equation (5) can be called the third-order d'Alembert-Lagrange principle of mechanico-electrical system.

Now, we write the principle (5) in parametric forms. The position vector $\mathbf{r}_i$ of the i-th particle and the capacitor charge E_k of the k-th loop can be, respectively, expressible as

$$\mathbf{r}_i = \mathbf{r}_i(q_\alpha, t) \quad (\alpha = 1, 2, \cdots, s), \; E_k = E_k(e_j, t) \quad (j = 1, 2, \cdots, m) \tag{7}$$

and the variationals of $\mathbf{r}_i$ and E_k can be given by

$$\delta \mathbf{r}_i = \sum_{\alpha=1}^{s} \frac{\partial \mathbf{r}_i}{\partial q_\alpha} \delta q_\alpha , \quad \delta E_k = \sum_{j=1}^{m} \frac{\partial E_k}{\partial e_j} \delta e_j \tag{8}$$

The first derivatives of $\frac{\partial \mathbf{r}_i}{\partial q_\alpha}$ and $\frac{\partial E_k}{\partial e_j}$ with respect to time t can be expressed by

$$\frac{d}{dt}\frac{\partial \mathbf{r}_i}{\partial q_\alpha} = \frac{\partial \dot{\mathbf{r}}_i}{\partial q_\alpha}, \frac{d}{dt}\frac{\partial E_k}{\partial e_j} = \frac{\partial \dot{E}_k}{\partial e_j} \tag{9}$$

From the relations (7), the second derivatives of $\mathbf{r}_i$ and E_k with respect to time t can be obtained. Taking the partial derivatives of $\ddot{\mathbf{r}}_i$ with respect to $\ddot{q}_\alpha$ and $\dot{q}_\alpha$, $\ddot{E}_k$ with respect to $\ddot{e}_j$ and $\dot{e}_j$ respectively, we have

$$\frac{\partial \ddot{\mathbf{r}}_i}{\partial \ddot{q}_\alpha} = \frac{\partial \mathbf{r}_i}{\partial q_\alpha}, \frac{\partial \ddot{\mathbf{r}}_i}{\partial \dot{q}_\alpha} = 2\frac{\partial \dot{\mathbf{r}}_i}{\partial q_\alpha}, \frac{\partial \ddot{E}_k}{\partial \ddot{e}_j} = \frac{\partial E_k}{\partial e_j}, \frac{\partial \ddot{E}_k}{\partial \dot{e}_j} = 2\frac{\partial \dot{E}_k}{\partial e_j} \tag{10}$$

Substituting Eqs.(8),(9) and (10) into principle (5), we obtain

$$\begin{aligned}&\sum_{\alpha=1}^{s}[-\frac{d}{dt}\frac{\partial}{\partial \ddot{q}_\alpha}(\frac{1}{2}\sum_{i=1}^{N} m_i \ddot{\mathbf{r}}_i \cdot \ddot{\mathbf{r}}_i) + \frac{1}{2}\frac{\partial}{\partial \dot{q}_\alpha}(\frac{1}{2}\sum_{i=1}^{N} m_i \ddot{\mathbf{r}}_i \cdot \ddot{\mathbf{r}}_i) + \sum_{i=1}^{N} \dot{\mathbf{F}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_\alpha}]\delta q_\alpha + \sum_{j=1}^{m}\left\{-\frac{d}{dt}\frac{\partial}{\partial \ddot{e}_j}(\frac{1}{2}\sum_{k=1}^{m+1}\sum_{l=1}^{m+1} L_{kl}\ddot{E}_l \ddot{E}_k)\right.\\
&-\frac{d}{dt}[\frac{\partial}{\partial \dot{e}_j}\sum_{\alpha=1}^{s}\frac{\partial}{\partial q_\alpha}(\frac{1}{2}\sum_{k=1}^{m+1}\sum_{l=1}^{m+1} L_{kl}\dot{E}_l \dot{E}_k)\dot{q}_\alpha] + \frac{1}{2}\frac{\partial}{\partial \dot{e}_j}(\frac{1}{2}\sum_{k=1}^{m+1}\sum_{l=1}^{m+1} L_{kl}\ddot{E}_l \ddot{E}_k) + \frac{\partial}{\partial e_j}\sum_{\alpha=1}^{s}\frac{\partial}{\partial q_\alpha}(\frac{1}{2}\sum_{k=1}^{m+1}\sum_{l=1}^{m+1} L_{kl}\dot{E}_l \dot{E}_k)\dot{q}_\alpha\\
&\left.-\frac{\partial}{\partial \ddot{e}_j}(\frac{1}{2}\sum_{k=1}^{m+1} R_k \ddot{E}_k^2) - \frac{\partial}{\partial \dot{e}_j}(\frac{1}{2}\sum_{k=1}^{m+1}\frac{\dot{E}_k^2}{C_k}) - \frac{\partial}{\partial e_j}\sum_{\alpha=1}^{s}\frac{\partial}{\partial q_\alpha}(\frac{1}{2}\sum_{k=1}^{m+1}\frac{E_k^2}{C_k})\dot{q}_\alpha + \sum_{k=1}^{m+1}\dot{U}_k \frac{\partial E_k}{\partial e_j}\right\}\delta e_j = 0\end{aligned} \tag{11}$$

where we notice that $L_{kl} = L_{kl}(q_\alpha)$, $C_k = C_k(q_\alpha)$, and suppose that each resistance in the circuit is linear, that is, R_k and $\dot{E}_k$ are independent.

Introducing the first velocity energy, the zero-order current energy (or the magnetic field energy), the first-order current energy, the zero-order electric field energy (or the electric field energy), the first-order electric field energy, the zero-order electric dissipative function, the first-order electric dissipative function of mechanico-electrical systems successively

$$S_1 = \frac{1}{2}\sum_{i=1}^{N} m_i \ddot{\mathbf{r}}_i \cdot \ddot{\mathbf{r}}_i \tag{12}$$

$$W_{m0} = \frac{1}{2}\sum_{k=1}^{m+1}\sum_{l=1}^{m+1} L_{kl}\dot{E}_l \dot{E}_k , W_{m1} = \frac{1}{2}\sum_{k=1}^{m+1}\sum_{l=1}^{m+1} L_{kl}\ddot{E}_l \ddot{E}_k \tag{13}$$

$$W_{e0} = \frac{1}{2}\sum_{k=1}^{m+1}\frac{E_k^2}{C_k}, W_{e1} = \frac{1}{2}\sum_{k=1}^{m+1}\frac{\dot{E}_k^2}{C_k} \tag{14}$$

$$F_{e0} = \frac{1}{2}\sum_{k=1}^{m+1} R_k \dot{E}_k^2 , F_{e1} = \frac{1}{2}\sum_{k=1}^{m+1} R_k \ddot{E}_k^2 \tag{15}$$

and introducing the time rate of change of generalized active forces and the time rate of change of generalized electromotive forces

$$Q_\alpha^1 = \sum_{i=1}^{N} \dot{\mathbf{F}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_\alpha} \quad (\alpha = 1, 2, \cdots, s) \tag{16}$$

$$U_j^1=\sum_{k=1}^{m+1}\dot{U}_k\frac{\partial E_k}{\partial e_j}\quad (j=1,2,\cdots,m) \tag{17}$$

Substituting Eqs.(12) -(17) into Eqs.(11) , principle (5) can be written as

$$\sum_{\alpha=1}^{s}(-\frac{d}{dt}\frac{\partial S_1}{\partial\ddot{q}_\alpha}+\frac{1}{2}\frac{\partial S_1}{\partial\dot{q}_\alpha}+Q_\alpha^1)\delta q_\alpha+\sum_{j=1}^{m}\{-\frac{d}{dt}\frac{\partial W_{m1}}{\partial\ddot{e}_j}+\frac{1}{2}\frac{\partial W_{m1}}{\partial\dot{e}_j}-\frac{d}{dt}(\frac{\partial}{\partial\dot{e}_j}\sum_{\alpha=1}^{s}\frac{\partial W_{m0}}{\partial q_\alpha}\dot{q}_\alpha)$$

$$+\frac{\partial}{\partial e_j}\sum_{\alpha=1}^{s}\frac{\partial(W_{m0}-W_{e0})}{\partial q_\alpha}\dot{q}_\alpha-\frac{\partial F_{e1}}{\partial\ddot{e}_j}-\frac{\partial W_{e1}}{\partial\dot{e}_j}+U_j^1\}\delta e_j=0 \tag{18}$$

Considering the mechanical part of mechanico-electrical systems is not only acted by the time rate of change of potential, dissipative and non-potential force, but also by the time rate of change of ponderomotive force, Eq.(16) can be expressed as follow

$$Q_\alpha^1=-\frac{\partial V_1}{\partial\dot{q}_\alpha}-\frac{\partial F_{m1}}{\partial\ddot{q}_\alpha}+Q_\alpha^{1\prime}+Q_\alpha^{1*}\quad (\alpha=1,2,\cdots,s) \tag{19}$$

where $V_1=V_1(q_\alpha,\dot{q}_\alpha)$ is the potential energy of time rate of change of conservative force, $F_{m1}=F_{m1}(q_\alpha,\dot{q}_\alpha,\ddot{q}_\alpha)$ is the first-order mechanical dissipative function, $Q_\alpha^{1\prime}=\sum_{i=1}^{N}\dot{\mathbf{F}}_i'\cdot\frac{\partial\mathbf{r}_i}{\partial q_\alpha}$ is the time rate of change of non-conservative generalized force which do not include mechanical dissipative force, Q_α^{1*} is the time rate of change of poderomotive force. Multiplying by $\dot{E}_k$ for both sides of Eq.(2), and summing over k, noticing Eqs.(13), (14) and $W_{m0}=W_{m0}(q_\alpha,\dot{e}_j)$, $W_{e0}=W_{e0}(q_\alpha,e_j)$, we obtain

$$\frac{dW_{m0}}{dt}+\sum_{\alpha=1}^{s}\frac{\partial W_{m0}}{\partial q_\alpha}\dot{q}_\alpha+\sum_{k=1}^{m+1}R_k\dot{E}_k^2+\frac{dW_{e0}}{dt}-\sum_{\alpha=1}^{s}\frac{\partial W_{e0}}{\partial q_\alpha}\dot{q}_\alpha-\sum_{k=1}^{m+1}U_k\dot{E}_k=0 \tag{20}$$

the energy balance equation of the system is

$$\sum_{k=1}^{m+1}U_k\dot{E}_k=\sum_{k=1}^{m+1}R_k\dot{E}_k^2+\frac{dW_{m0}}{dt}+\frac{dW_{e0}}{dt}+\sum_{\alpha=1}^{s}Q_\alpha^*\dot{q}_\alpha \tag{21}$$

substituting Eq.(21) into Eq.(20), the pondermotive force can be expressed as follow

$$Q_\alpha^*=\frac{\partial W_{m0}}{\partial q_\alpha}-\frac{\partial W_{e0}}{\partial q_\alpha}=\frac{\partial}{\partial q_\alpha}(W_{m0}-W_{e0})\quad (\alpha=1,2,\cdots,s) \tag{22}$$

From Eq.(22), we have

$$Q_\alpha^{1*}=\frac{d}{dt}\frac{\partial}{\partial q_\alpha}(W_{m0}-W_{e0})=\frac{\partial}{\partial q_\alpha}(\dot{W}_{m0}-\dot{W}_{e0})\quad (\alpha=1,2,\cdots,s) \tag{23}$$

Substituting Eq.(23) into Eq.(19), we obtain

$$Q_\alpha^1=-\frac{\partial V_1}{\partial\dot{q}_\alpha}-\frac{\partial F_{m1}}{\partial\ddot{q}_\alpha}+Q_\alpha^{1\prime}+\frac{\partial}{\partial q_\alpha}(\dot{W}_{m0}-\dot{W}_{e0})\quad (\alpha=1,2,\cdots,s) \tag{24}$$

Thus, Eq.(18) can be written as

$$\sum_{\alpha=1}^{s}[-\frac{d}{dt}\frac{\partial S_1}{\partial\ddot{q}_\alpha}+\frac{1}{2}\frac{\partial S_1}{\partial\dot{q}_\alpha}-\frac{\partial V_1}{\partial\dot{q}_\alpha}-\frac{\partial F_{m1}}{\partial\ddot{q}_\alpha}+\frac{\partial}{\partial q_\alpha}(\dot{W}_{m0}-\dot{W}_{e0})+Q_\alpha^{1\prime}]\delta q_\alpha+\sum_{j=1}^{m}\{-\frac{d}{dt}\frac{\partial W_{m1}}{\partial\ddot{e}_j}+\frac{1}{2}\frac{\partial W_{m1}}{\partial\dot{e}_j}$$

$$-\frac{d}{dt}\frac{\partial}{\partial\dot{e}_j}\sum_{\alpha=1}^{s}\frac{\partial W_{m0}}{\partial q_\alpha}\dot{q}_\alpha+\frac{\partial}{\partial e_j}\sum_{\alpha=1}^{s}\frac{\partial(W_{m0}-W_{e0})}{\partial q_\alpha}\dot{q}_\alpha-\frac{\partial F_{e1}}{\partial\ddot{e}_j}-\frac{\partial W_{e1}}{\partial\dot{e}_j}+U_j^1\}\delta e_j=0 \quad (25)$$

Equation (25) is the parametric form of Euler-Lagrange for the third-order d'Alembert-Lagrange principle of mechanico-electrical systems.

Introducing the first-order Lagrange function of mechanico-electrical systems

$$L_1=L_1(q_\alpha,\dot{q}_\alpha,\ddot{q}_\alpha,e_j,\dot{e}_j,\ddot{e}_j)=S_1(q_\alpha,\dot{q}_\alpha,\ddot{q}_\alpha)-2V_1(q_\alpha,\dot{q}_\alpha)+W_{m1}(q_\alpha,\dot{e}_j,\ddot{e}_j)-2W_{e1}(q_\alpha,e_j,\dot{e}_j) \quad (26)$$

the principle (25) can also be written as

$$\sum_{\alpha=1}^{s}[-\frac{d}{dt}\frac{\partial L_1}{\partial\ddot{q}_\alpha}+\frac{1}{2}\frac{\partial L_1}{\partial\dot{q}_\alpha}-\frac{\partial F_{m1}}{\partial\ddot{q}_\alpha}+\frac{\partial}{\partial q_\alpha}(\dot{W}_{m0}-\dot{W}_{e0})+Q_\alpha^{1\prime}]\delta q_\alpha+\sum_{j=1}^{m}\{-\frac{d}{dt}\frac{\partial L_1}{\partial\ddot{e}_j}+\frac{1}{2}\frac{\partial L_1}{\partial\dot{e}_j}$$

$$-\frac{d}{dt}[\frac{\partial}{\partial\dot{e}_j}\sum_{\alpha=1}^{s}\frac{\partial W_{m0}}{\partial q_\alpha}\dot{q}_\alpha]+\frac{\partial}{\partial e_j}\sum_{\alpha=1}^{s}\frac{\partial(W_{m0}-W_{e0})}{\partial q_\alpha}\dot{q}_\alpha-\frac{\partial F_{e1}}{\partial\ddot{e}_j}+U_j^1\}\delta e_j=0 \quad (27)$$

Introducing the first-order dissipative function of mechanico-electrical systems which equals to the sum of the first-order electrical and mechanical dissipative function, that is

$$F_1=F_{e1}(\dot{e}_j,\ddot{e}_j)+F_{m1}(q_\alpha,\dot{q}_\alpha,\ddot{q}_\alpha) \quad (28)$$

then, the principle (27) can also be expressed as the unified form

$$\sum_{\alpha=1}^{s}[-\frac{d}{dt}\frac{\partial L_1}{\partial\ddot{q}_\alpha}+\frac{1}{2}\frac{\partial L_1}{\partial\dot{q}_\alpha}-\frac{\partial F_1}{\partial\ddot{q}_\alpha}+\frac{\partial}{\partial q_\alpha}(\dot{W}_{m0}-\dot{W}_{e0})+Q_\alpha^{1\prime}]\delta q_\alpha+\sum_{j=1}^{m}\{-\frac{d}{dt}\frac{\partial L_1}{\partial\ddot{e}_j}+\frac{1}{2}\frac{\partial L_1}{\partial\dot{e}_j}$$

$$-\frac{d}{dt}[\frac{\partial}{\partial\dot{e}_j}\sum_{\alpha=1}^{s}\frac{\partial W_{m0}}{\partial q_\alpha}\dot{q}_\alpha]+\frac{\partial}{\partial e_j}\sum_{\alpha=1}^{s}\frac{\partial(W_{m0}-W_{e0})}{\partial q_\alpha}\dot{q}_\alpha-\frac{\partial F_1}{\partial\ddot{e}_j}+U_j^1\}\delta e_j=0 \quad (29)$$

From the relations (7) and (8), the third-order derivatives of $\mathbf{r}_i$ and E_k with respect to time t can be obtained. The partial derivatives of $\dddot{\mathbf{r}}_i$ with respect to $\dddot{q}_\alpha$ and $\ddot{q}_\alpha$, and $\dddot{E}_k$ with respect to $\ddot{e}_j$ and $\ddot{e}_j$ can be, respectively, given by

$$\frac{\partial\dddot{\mathbf{r}}_i}{\partial\dddot{q}_\alpha}=\frac{\partial\mathbf{r}_i}{\partial q_\alpha},\frac{\partial\dddot{\mathbf{r}}_i}{\partial\ddot{q}_\alpha}=3\frac{\partial\dot{\mathbf{r}}_i}{\partial q_\alpha},\quad\frac{\partial\dddot{E}_k}{\partial\dddot{e}_j}=\frac{\partial E_k}{\partial e_j},\quad\frac{\partial\dddot{E}_k}{\partial\ddot{e}_j}=3\frac{\partial\dot{E}_k}{\partial e_j} \quad (30)$$

By utilizing Eqs.(8), (10), (13)-(17), (24) and (30), the principle (5) can also written as

$$\sum_{\alpha=1}^{s}[-\frac{\partial\dot{S}_1}{\partial\ddot{q}_\alpha}+\frac{3}{2}\frac{\partial S_1}{\partial\dot{q}_\alpha}-\frac{\partial V_1}{\partial\dot{q}_\alpha}-\frac{\partial F_{m1}}{\partial\ddot{q}_\alpha}+Q_\alpha^{1\prime}+\frac{\partial}{\partial q_\alpha}(\dot{W}_{m0}-\dot{W}_{e0})]\delta q_\alpha$$

$$+\sum_{j=1}^{m}[-\frac{\partial}{\partial\ddot{e}_j}\frac{dW_{m1}}{dt}-\frac{\partial}{\partial\dot{e}_j}\frac{d}{dt}(\sum_{\alpha=1}^{s}\frac{\partial W_{m0}}{\partial q_\alpha}\dot{q}_\alpha)-\frac{\partial F_{e1}}{\partial\ddot{e}_j}-\frac{\partial W_{e1}}{\partial\dot{e}_j}-\frac{\partial}{\partial e_j}\sum_{\alpha=1}^{s}\frac{\partial W_{e0}}{\partial q_\alpha}\dot{q}_\alpha+U_k^1]\delta e_j=0 \quad (31)$$

Equation (31) is the parametric form of Nielsen for the third-order d'Alembert-Lagrange principle of mechanico-electrical systems.

Introducing the second-order velocity energy and the second-order current energy of mechanico-electrical systems successively

$$S_2=\frac{1}{2}\sum_{i=1}^{N}m_i\dddot{\mathbf{r}}_i\cdot\dddot{\mathbf{r}}_i,W_{m2}=\frac{1}{2}\sum_{k=1}^{m+1}\sum_{l=1}^{m+1}L_{kl}\dddot{E}_k\dddot{E}_l \quad (32)$$

the principle (5) can also be written as

$$\sum_{\alpha=1}^{s}[-\frac{\partial S_2}{\partial \dddot{q}_\alpha}-\frac{\partial V_1}{\partial \dot{q}_\alpha}-\frac{\partial F_{m1}}{\partial \ddot{q}_\alpha}+Q_\alpha^{1\prime}+\frac{\partial}{\partial q_\alpha}(\dot{W}_{m0}-\dot{W}_{e0})]\delta q_\alpha+\sum_{j=1}^{m}[-\frac{\partial W_{m2}}{\partial \dddot{e}_j}-2\frac{\partial}{\partial \ddot{e}_j}(\sum_{\alpha=1}^{s}\frac{\partial W_{m1}}{\partial q_\alpha}\dot{q}_\alpha)$$

$$-(\sum_{\alpha=1}^{s}\sum_{\beta=1}^{s}\frac{\partial^2 W_{m0}}{\partial q_\alpha \partial q_\beta}\dot{q}_\alpha\dot{q}_\beta+\sum_{\alpha=1}^{s}\frac{\partial W_{m0}}{\partial q_\alpha}\ddot{q}_\alpha)-\frac{\partial F_{e1}}{\partial \ddot{e}_j}-\frac{\partial W_{e1}}{\partial \dot{e}_j}-\frac{\partial}{\partial e_j}\sum_{\alpha=1}^{s}\frac{\partial W_{e0}}{\partial q_\alpha}\dot{q}_\alpha+U_j^1]\delta e_j=0 \tag{33}$$

Equation (33) is the parametric form of Appell for the third-order d'Alembert-Lagrange principle of mechanico-electrical systems.

3. Third-Order Differential Equations of Motion for Holonomic Mechanico-Electrical Systems

For the holonomic mechanico-electrical systems, because δq_α $(\alpha=1,2,\cdots,s)$ and δe_j $(j=1,2,\cdots,m)$ are independent of each other, thus by Eq.(25), we have

$$\frac{d}{dt}\frac{\partial S_1}{\partial \ddot{q}_\alpha}-\frac{1}{2}\frac{\partial S_1}{\partial \dot{q}_\alpha}=-\frac{\partial V_1}{\partial \dot{q}_\alpha}-\frac{\partial F_{m1}}{\partial \ddot{q}_\alpha}+\frac{\partial}{\partial q_\alpha}(\dot{W}_{m0}-\dot{W}_{e0})+Q_\alpha^{1\prime}\quad(\alpha=1,2,\cdots,s) \tag{34}$$

$$\frac{d}{dt}\frac{\partial W_{m1}}{\partial \ddot{e}_j}-\frac{1}{2}\frac{\partial W_{m1}}{\partial \dot{e}_j}+\frac{d}{dt}\frac{\partial}{\partial \dot{e}_j}\sum_{\alpha=1}^{s}\frac{\partial W_{m0}}{\partial q_\alpha}\dot{q}_\alpha-\frac{\partial}{\partial e_j}\sum_{\alpha=1}^{s}\frac{\partial(W_{m0}-W_{e0})}{\partial q_\alpha}\dot{q}_\alpha+\frac{\partial F_{e1}}{\partial \ddot{e}_j}+\frac{\partial W_{e1}}{\partial \dot{e}_j}=U_j^1\quad(j=1,2,\cdots,m) \tag{35}$$

or by Eq.(29), we have

$$\frac{d}{dt}\frac{\partial L_1}{\partial \ddot{q}_\alpha}-\frac{1}{2}\frac{\partial L_1}{\partial \dot{q}_\alpha}=-\frac{\partial F_1}{\partial \ddot{q}_\alpha}+\frac{\partial}{\partial q_\alpha}(\dot{W}_{m0}-\dot{W}_{e0})+Q_\alpha^{1\prime}\quad(\alpha=1,2,\cdots,s) \tag{36}$$

$$\frac{d}{dt}\frac{\partial L_1}{\partial \ddot{e}_j}-\frac{1}{2}\frac{\partial L_1}{\partial \dot{e}_j}+\frac{d}{dt}[\frac{\partial}{\partial \dot{e}_j}\sum_{\alpha=1}^{s}\frac{\partial W_{m0}}{\partial q_\alpha}\dot{q}_\alpha]-\frac{\partial}{\partial e_j}\sum_{\alpha=1}^{s}\frac{\partial(W_{m0}-W_{e0})}{\partial q_\alpha}\dot{q}_\alpha+\frac{\partial F_1}{\partial \ddot{e}_j}=U_j^1\quad(j=1,2,\cdots,m) \tag{37}$$

Equations (34) and (35) or Equations (36) and (37) can be called the third-order Euler-Lagrange equations of holonomic mechanico-electrical systems.

By Eq.(31), we obtain

$$\frac{\partial \dot{S}_1}{\partial \dddot{q}_\alpha}-\frac{3}{2}\frac{\partial S_1}{\partial \dot{q}_\alpha}=-\frac{\partial V_1}{\partial \dot{q}_\alpha}-\frac{\partial F_{m1}}{\partial \ddot{q}_\alpha}+Q_\alpha^{1\prime}+\frac{\partial}{\partial q_\alpha}(\dot{W}_{m0}-\dot{W}_{e0})\quad(\alpha=1,2,\cdots,s) \tag{38}$$

$$\frac{\partial}{\partial \dddot{e}_j}\frac{dW_{m1}}{dt}+\frac{\partial}{\partial \dot{e}_j}\frac{d}{dt}(\sum_{\alpha=1}^{s}\frac{\partial W_{m0}}{\partial q_\alpha}\dot{q}_\alpha)+\frac{\partial F_{e1}}{\partial \ddot{e}_j}+\frac{\partial W_{e1}}{\partial \dot{e}_j}+\frac{\partial}{\partial e_j}\sum_{\alpha=1}^{s}\frac{\partial W_{e0}}{\partial q_\alpha}\dot{q}_\alpha=U_k^1\quad(j=1,2,\cdots,m) \tag{39}$$

Equations (38) and (39) can be called the third-order Nielsen equations of holonomic mechanico-electrical systems.

By Eq. (33), we obtain

$$\frac{\partial S_2}{\partial \dddot{q}_\alpha}=-\frac{\partial V_1}{\partial \dot{q}_\alpha}-\frac{\partial F_{m1}}{\partial \ddot{q}_\alpha}+Q_\alpha^{1\prime}+\frac{\partial}{\partial q_\alpha}(\dot{W}_{m0}-\dot{W}_{e0})\quad(\alpha=1,2,\cdots,s) \tag{40}$$

$$\frac{\partial W_{m2}}{\partial \dddot{e}_j}+2\frac{\partial}{\partial \ddot{e}_j}(\sum_{\alpha=1}^{s}\frac{\partial W_{m1}}{\partial q_\alpha}\dot{q}_\alpha)+\sum_{\alpha=1}^{s}\sum_{\beta=1}^{s}\frac{\partial^2 W_{m0}}{\partial q_\alpha \partial q_\beta}\dot{q}_\alpha\dot{q}_\beta+\sum_{\alpha=1}^{s}\frac{\partial W_{m0}}{\partial q_\alpha}\ddot{q}_\alpha$$

$$+\frac{\partial F_{e1}}{\partial \ddot{e}_j}+\frac{\partial W_{e1}}{\partial \dot{e}_j}+\frac{\partial}{\partial e_j}\sum_{\alpha=1}^{s}\frac{\partial W_{e0}}{\partial q_\alpha}\dot{q}_\alpha=U_j^1\quad(j=1,2,\cdots,m) \tag{41}$$

Equations (40) and (41) can be called the third-order Appell equations of holonomic mechanico-electrical systems.

4. Conclusion

In this paper, we have established the third-order d'Alembert-Lagrange principle of mechanico-electrical systems and its parametric forms, and obtained the third-order differential equations of holonomic mechanico-electrical systems. The third-order differential variational principles and differential equations of mechanico-electrical systems not only is a supplement to the differential variational principles and differential equations of mechanico-electrical systems, but also is an extension of the theory of the third-order differential variational principles and differential equations.

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